J. Math. Anal. Appl. 382 (2011) 148–161

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Counter-examples of regularity behavior for σ-evolution equations

Ziheng Tu [a,*], Xiaojun Lu [b,1]

[a] Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
[b] Basque Center for Applied Mathematics (BCAM), Bizkaia Technology Park, Building 500, E-48160 Derio, Spain

ARTICLE INFO

Article history:
Received 3 July 2010
Available online 23 April 2011
Submitted by M. Nakao

Keywords:
μ-Loss of derivatives
Principle symbols
Floquet theory
Instability argument
σ-Evolution model

ABSTRACT

In this paper, we mainly discuss the infinite loss of regularity and μ-loss for a σ-evolution type model with oscillating in time coefficients. On the one hand, an explicit counter-example has been constructed in the frequency space to show the precise infinite loss of regularity. On the other hand, due to the finite propagation speed property for $\sigma \in (0,1]$, we construct the counter-example of a sequence of solutions in $\mathbb{R}$ by applying state of art techniques.

1. Introduction

Loss of regularity is a very popular topic in the theoretical analysis. For instance, there is no loss of regularity for the typical wave equation due to the energy conservation law. While for a general strictly hyperbolic operator with oscillating in time coefficients for the principal elliptical part, the loss appears, see [13]. A similar conclusion follows for the weakly hyperbolic operators with the same kind of coefficients, see [9]. In 2008, D. Fang, X. Lu and M. Reissig generalized the models in [2] and [13] by

$$\mathscr{L} = \partial_t^2 + A_0(t, D_x) + A_1(t, D_x) + A_2(t, D_x),$$

where $A_0(t, D_x) = b(t)(-\Delta)^\sigma$, $\sigma \in \mathbb{R}_+$ and $A_i(t, D_x)$, $i = 0, 1, 2$, are linear pseudodifferential operators defined on the temping distribution space $\mathcal{S}'$ by

$$A_i(t, D_x)u = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \exp\bigl(i\langle x - y, \xi\rangle\bigr) A_i(t, \xi) u(y)\, dy\, d\xi.$$

The coupled influence from the principal part $(-\Delta)^\sigma$ and its oscillating coefficient $b(t)$ are clearly observed in the expression of loss. In [6], the authors mainly discussed the optimality of the finite loss by applying the spectral theory of the operator $(-\Delta)^\sigma$ on the torus and extended the instability arguments in [2–5] to Fourier multipliers. In this regard, please see [6] for more details. In contrast, we consider in this paper the optimality of infinite loss by constructing an explicit counter-example by virtue of the *Floquet theory*. Moreover, a genuine counter-example of a sequence of solutions will be constructed due to the finite propagation speed property for $\sigma \in (0,1]$. And we have refined the estimates in [13,6] and generalized the discussion in [8].

* Corresponding author.
E-mail addresses: tuziheng@163.com (Z. Tu), lvxiaojun1119@hotmail.de (X. Lu).

[1] This work belongs to the following projects: 246775-NUMERIWAVES funded by European Research Council (ERC), MTM2008-03541, funded by MICIN, ESF Research Networking Programme OPTPDE, PI2010-04, funded by Basque Government.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.04.036

The technique of *Floquet theory* arises from [16] in treating the weakly hyperbolic problem with infinitely degenerating coefficients. In [14,15], the authors extended this technique to strictly hyperbolic cases. Recently, this method has been further applied to the discussion of p-evolution type models in [5]. All these counter-examples are constructed with respect to the measure function $\mu(t) = (\log(1/t))^{q-1}$, $q > 2$. As is shown in Section 2, $\mu(t) = \log(1/t)$ is the critical case for C^∞ well-posedness. Here we construct a more delicate counter-example, which is arbitrarily close to the critical case with

$$\mu(t) = \log(1/t)\log^{[n]}(1/t), \quad n \geqslant 2. \tag{1.1}$$

By the application of *Floquet theory*, we can show that the infinite loss of derivatives really appears. Now we state our main results.

Theorem 1.1. *Consider the Cauchy problem of the following wave equation,*

$$\begin{cases} u_{tt} + a(t)(-\triangle)^\sigma u = 0, \\ u(0,x) = \varphi(x), \qquad u_t(0,x) = \psi(x) \end{cases} \tag{1.2}$$

on $[0,T] \times \mathbb{R}^N$*. Assume that the initial data* $\varphi,\ \psi$ *belong to* $H^s,\ H^{s-\sigma}$ *respectively, the coefficient* $a(t)$ *belongs to* $L^\infty \cap C^2(0,T]$ *and its first and second order derivatives satisfy the condition*

$$\left|a^{(k)}(t)\right| \leqslant C_k\left(\frac{1}{t}\mu(t)\right)^k, \quad k = 1, 2,$$

where $\mu(t)$ *is a continuous positive decreasing function. Then the problem has a unique solution* u *belonging to the following function spaces*:

$$u \in C\left([0,T); \exp\left(C_1\mu\left(F^{-1}\left(\frac{\langle D_x\rangle^\sigma}{N}\right)\right)\right)H^s\left(\mathbb{R}^N\right)\right),$$

$$u_t \in C\left([0,T]; \exp\left(C_1\mu\left(F^{-1}\left(\frac{\langle D_x\rangle^\sigma}{N}\right)\right)\right)H^{s-\sigma}\left(\mathbb{R}^N\right)\right),$$

where $F(t) := \frac{\mu(t)}{t}$ *and* F^{-1} *is its inverse function.*

Remark 1.1. The above expression is kind of complicated, the interested readers can refer to [6] for specific examples.

Furthermore, we restrict to the following σ-evolution equation

$$\partial_t^2 u + b^2\left(\left(\log(1/t)\right)^2 \log^{[n]}(1/t)\right)(-\Delta)^\sigma u = 0, \tag{1.3}$$

where $n \geqslant 2$ and $b = b(s)$ is a positive, 1-periodic, non-constant function belonging to $C^2(\mathbb{R})$. One can check that all the assumptions in Theorem 1.1 are satisfied with $\mu(t)$ stated in (1.1). We have the following result in the ultradistributional sense.

Theorem 1.2. *Under the above assumptions for* $b(t)$*, when* $n \geqslant 2$*, then the Cauchy problem of* (1.3) *is not* H^∞ *well posed, that is, there exists an infinite loss of derivatives.*

Actually, the operator $\mathscr{L} = \partial_t^2 + (-\Delta)^\sigma$ has the finite speed of propagation if and only if the equation is of Kowalewski type, namely $\sigma \in (0,1]$. In this respect, the interested readers are referred to [1,7,12] for precise proof. For $\sigma > 1$, propagation of supports has no meaning. As a result, we construct a counter-example in $[0,T] \times \mathbb{R}$ with $\sigma \in (0,1]$. We consider the following strictly hyperbolic Cauchy problem in $[0,T] \times \mathbb{R}$

$$\partial_t^2 u + b(t)\left(-\partial_x^2\right)^\sigma u = 0, \qquad u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x).$$

Due to the finite propagation of support when $\sigma \in (0,1]$, we have the following result.

Theorem 1.3. *For the above type of Cauchy problems, there exist a sequence of coefficients* $\{b_k(t)\}_k$ *satisfying all assumptions of Theorem* 1.1 *and a sequence of data* $\{(u_{0,k}(x), u_{1,k}(x))\}_k \in H^s(\mathbb{R}) \times H^{s-\sigma}(\mathbb{R})$ *in instability intervals* $\{I_k\}_k$*,* $s \in \mathbb{R}_+$*. Correspondingly, we can find two zero sequences* $\{t_k^{(1)}\}_k,\ \{t_k^{(2)}\}_k$*, such that the following estimates hold for the sequence of solutions* $\{u_k(t,x)\}_k$:

$$\left\|\exp\left(-c_1\mu\left(F^{-1}\left(\langle D_x\rangle^\sigma/2^P\right)\right)\right)u_k\left(t_k^{(2)},\cdot\right)\right\|_{H^s(\mathbb{R})} \geqslant C_k\left\|u_k\left(t_k^{(1)},\cdot\right)\right\|_{H^s(\mathbb{R})},$$

where P *is positive integer, the positive constant* c_1 *is independent of* k *and* $\sup_k C_k = +\infty$.

The rest of the paper is organized as follows. Section 2 is a sketch of proof of Theorem 1.1 for the sake of self-containedness. The interested readers can refer to [6] for details. In this proof, some powerful tools from micro-local analysis and WKB analysis will be used to obtain the precise μ-loss of derivatives. In Section 3, we discuss the optimality of our statement concerning the infinite loss of regularity by the application of *Floquet theory*. In Section 4, we will give a detailed construction process for general μ-loss by instability argument.

2. Proof of Theorem 1.1

Applying Fourier-transform on (1.2) gives,

$$\begin{cases} \hat{u}_{tt} + a(t)|\xi|^{2\sigma}\hat{u} = 0, \\ \hat{u}(0,\xi) = \hat{\varphi}(\xi), \qquad \hat{u}_t(0,\xi) = \hat{\psi}(\xi). \end{cases} \tag{2.1}$$

We divide the phase space into three parts as following:

$$Z_{low}(M) = \{(t,\xi) \in [0,T] \times \{|\xi| \leqslant M\}\};$$
$$Z_{pd}(M,N) = \{(t,\xi) \in [0,T] \times \{|\xi| \geqslant M\}\colon t\langle\xi\rangle^\sigma \leqslant N\mu(t)\};$$
$$Z_{hyp}(M,N) = \{(t,\xi) \in [0,T] \times \{|\xi| \geqslant M\}\colon t\langle\xi\rangle^\sigma \geqslant N\mu(t)\}.$$

Here M is a sufficiently large positive constant. $Z_{pd}(M,N)$ and $Z_{hyp}(M,N)$ are two zones divided by the separating line $\{(t_\xi,\xi)\colon \frac{\mu(t_\xi)}{t_\xi} = \frac{\langle\xi\rangle^\sigma}{N}\}$. Next we estimate each zone.

2.1. Estimations in zones $Z_{low}(M)$ and $Z_{pd}(M,N)$

In the zone $Z_{low}(M)$, since the frequency is low, the problem is easy to solve. Introducing the micro-energy $V = (|\xi|^\sigma\hat{u}, D_t\hat{u})^T$, Eq. (2.1) is equivalent to the following system,

$$D_tV = \begin{pmatrix} 0 & |\xi|^\sigma \\ a(t)|\xi|^\sigma & 0 \end{pmatrix} V := A(t,\xi)V. \tag{2.2}$$

The associated fundamental solution X satisfies

$$V(t,\xi) := X(t,s,\xi)V(s,\xi), \quad X(r,r,\xi) = I,$$

which has the following explicit representation,

$$X(t,r,\xi) = I + \sum_{k=1}^{\infty} i^k \int_r^t A(t_1,\xi) \int_r^{t_1} A(t_2,\xi) \cdots \int_r^{t_{k-1}} A(t_k,\xi)\,dt_k \cdots dt_1. \tag{2.3}$$

Lemma 2.1. *For $k \in \mathbb{N}_+$ it holds*

$$\left\| \int_r^t A(t_1,\xi) \int_r^{t_1} A(t_2,\xi) \cdots \int_r^{t_{k-1}} A(t_k,\xi)\,dt_k \cdots dt_1 \right\| \leqslant \frac{1}{k!}\left(\int_s^t \|A(r,\xi)\|\,dr \right)^k.$$

Proof. In fact,

$$\int_s^t \|A(t_1,\xi)\| \int_s^{t_1} \|A(t_2,\xi)\|\,dt_2\,dt_1 = \int_s^t \frac{\partial}{\partial t_1}\left(\int_s^{t_1} \|A(t_2,\xi)\|\,dt_2 \right)\left(\int_s^{t_1} \|A(t_2,\xi)\|\,dt_2 \right) dt_1$$
$$= \int_s^t \frac{1}{2}\frac{\partial}{\partial t_1}\left(\int_s^{t_1} \|A(t_2,\xi)\|\,dt_2 \right)^2 dt_1 = \frac{1}{2}\left(\int_s^t \|A(r,\xi)\|\,dr \right)^2.$$

This lemma follows immediately by the induction method. □

This lemma implies

$$\|X\| \leqslant 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\int_0^T \|A(t,\xi)\|\,dt \right)^k,$$

and for low frequencies,

$$\|A(t,\xi)\| \leqslant C|\xi|^\sigma \leqslant C(M) \quad \text{and} \quad \int_0^t \|A(s,\xi)\|\,ds \leqslant C(M)T, \quad \forall t \in [0,T].$$

Thus

$$\|X\| \leqslant \exp\bigl(C(M)T\bigr),$$

consequently

$$\bigl|V(t,\xi)\bigr| \leqslant C(M,T)\bigl|V_0(\xi)\bigr|. \tag{2.4}$$

Next we consider the zone $Z_{pd}(M)$. The estimation is almost the same as in $Z_{low}(M)$ with some minor modification. Using the same micro-energy V similarly, we have

$$\|X\| \leqslant 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\int_0^{t_\xi} \|A(t,\xi)\|\,dt\right)^k.$$

By the definition of zone Z_{pd},

$$\|A(t,\xi)\| \leqslant C\langle\xi\rangle^\sigma \quad \text{and} \quad t_\xi\langle\xi\rangle^\sigma = M\mu(t_\xi).$$

Thus

$$\|X(t,r,\xi)\| \leqslant 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\bigl(CM\mu(t_\xi)\bigr)^k \leqslant \exp\bigl(CM\mu(t_\xi)\bigr),$$

which implies

$$\bigl|V(t,\xi)\bigr| \leqslant \exp\bigl(CM\mu(t_\xi)\bigr)\bigl|V_0(\xi)\bigr| \quad \forall (t,\xi) \in Z_{pd}. \tag{2.5}$$

2.2. Diagonalization in $Z_{hyp}(M,N)$ and estimation

In $Z_{hyp}(M,N)$, the situation is completely different from the previous two cases. The frequencies are not bounded any more. First we introduce the symbol classes. Given $m_1 \in \mathbb{R}$, $m_2 \geqslant 0$, $r \leqslant 2$, $r \in \mathbb{N}$ we define:

$$S_r\{m_1,m_2\} := \left\{ d = d(t,\xi) \in L^\infty\bigl([0,T]\times\mathbb{R}^N\bigr) \;\Big|\; \bigl|D_t^k D_\xi^\alpha d(t,\xi)\bigr| \leqslant C_{k,\alpha}\langle\xi\rangle^{m_1-|\alpha|}\left(\frac{1}{t}\mu(t)\right)^{m_2+k}, \right.$$
$$\left. k \leqslant r,\ \alpha \in \mathbb{Z}_+^N,\ (t,\xi) \in Z_{hyp}(M,N) \right\}.$$

A simple calculation indicates that $S_r\{m_1,\ m_2\}$ has the following properties:

- $S_{r+1}\{m_1,m_2\} \subset S_r\{m_1,m_2\}$;
- $S_r\{m_1-p,m_2\} \subset S_r\{m_1,m_2\}$;
- $S_r\{m_1-\sigma p,m_2+p\} \subset S_r\{m_1,m_2\}$ where $p \geqslant 0$;
- if $a \in S_r\{m_1,m_2\}$ and $b \in S_r\{k_1,k_2\}$, then $ab \in S_r\{m_1+k_1,m_2+k_2\}$;
- if $a \in S_r\{m_1,m_2\}$, then $D_t a \in S_{r-1}\{m_1,m_2+1\}$, and $D_\xi^\alpha a \in S_r\{m_1-|a|,m_2\}$.

Let $V = (\sqrt{a(t)}|\xi|^\sigma\hat{u}, D_t\hat{u})^T$ be the micro-energy and we have the following first order system:

$$D_t V - \begin{pmatrix} 0 & \sqrt{a(t)}|\xi|^\sigma \\ \sqrt{a(t)}|\xi|^\sigma & 0 \end{pmatrix} V - \frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V = 0. \tag{2.6}$$

Let $V_0 := \mathcal{M}V$, $\mathcal{M} := \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. The above system can be transformed into the first order system below:

$$D_t V_0 - \mathcal{M}\begin{pmatrix} 0 & \sqrt{a(t)}|\xi|^\sigma \\ \sqrt{a(t)}|\xi|^\sigma & 0 \end{pmatrix}\mathcal{M}^{-1}V_0 - \mathcal{M}\frac{D_t a}{2a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathcal{M}^{-1}V_0 = 0,$$
$$D_t V_0 - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_0 - \frac{D_t a}{4a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0 = 0,$$

where $\tau_{1/2} := \mp\sqrt{a(t)}|\xi|^\sigma + \frac{1}{4}\frac{D_t a}{a}$. Now we simplify this system in the form

$$D_t V_0 - \mathcal{D} V_0 - R_0 V_0 = 0,$$

where

$$\mathcal{D} := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in S_1\{\sigma, 0\}; \qquad R_0 = \frac{1}{4}\frac{D_t a}{a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in S_1\{0, 1\}.$$

Set

$$\mathcal{N}^{(1)} := -\frac{1}{4}\frac{D_t a}{a}\begin{pmatrix} 0 & \frac{1}{\tau_1 - \tau_2} \\ \frac{1}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \frac{D_t a}{8a^{3/2}|\xi|^\sigma}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\mathcal{N}_1 := \mathcal{N}^{(1)} + I.$$

$\mathcal{N}_1$ is invertible in $Z_{hyp}(N)$ for sufficiently large N, since

$$\|\mathcal{N}^{(1)}\| \leqslant \left\| \frac{D_t a}{8a^{3/2}|\xi|^\sigma} \right\| \leqslant \frac{\mu(t)}{|\xi|^\sigma t} \leqslant \frac{\mu(t)}{t\langle\xi\rangle^\sigma} = \frac{1}{N} < \frac{1}{2}.$$

The second inequality holds since $D_t a \in S_1\{0, 1\}$.

Observe that

$$(D_t - \mathcal{D} - R_0)\mathcal{N}_1 = \mathcal{N}_1(D_t - \mathcal{D} - R_1),$$

where $R_1 := -\mathcal{N}_1^{-1}(D_t\mathcal{N}^{(1)} - R_0\mathcal{N}^{(1)})$. Notice that

$$\mathcal{N}^{(1)} \in S_1\{-\sigma, 1\} \subset S_1\{0, 0\}, \qquad \mathcal{N}_1 \in S_1\{0, 0\} \quad \text{and} \quad R_1 \in S_0\{-\sigma, 2\},$$

the transformation $V_0 =: \mathcal{N}_1 V_1$ gives the following first order system:

$$D_t V_1 - \mathcal{D} V_1 - R_1 V_1 = 0, \quad \mathcal{D} \in S_1\{\sigma, 0\},\ R_1 \in S_0\{-\sigma, 2\}.$$

This is the second step of diagonalization of our system (2.6) modulo $R_1 \in S_0\{-\sigma, 2\}$.

Consider the system

$$D_t V_1 - \mathcal{D} V_1 - R_1 V_1 = 0, \tag{2.7}$$

with the initial condition

$$V_1(t_\xi, \xi) = V_{1,0}(\xi) := \mathcal{N}_1^{-1}(t_\xi, \xi)\mathcal{M} V(t_\xi, \xi).$$

It is clear that $V = \mathcal{M}^{-1}\mathcal{N}_1 V_1$ is the solution of system (2.6) in $Z_{hyp}(M, N)$.

It is easy to verify that the matrix-valued function

$$E_2(t, r, \xi) := \begin{pmatrix} \exp(i\int_r^t (-\sqrt{a(s)}|\xi|^\sigma + \frac{D_s a(s)}{4a(s)})\, ds) & 0 \\ 0 & \exp(i\int_r^t (\sqrt{a(s)}|\xi|^\sigma + \frac{D_s a(s)}{4a(s)})\, ds) \end{pmatrix}$$

solves the Cauchy problem $(D_t - \mathcal{D})E(t, r, \xi) = 0$, $E(r, t, \xi) = I$. Define

$$H(t, r, \xi) := E_2(r, t, \xi) R_1(t, \xi) E_2(t, r, \xi).$$

The fact

$$\int_r^t \frac{\partial_s a(s)}{4a(s)}\, ds = \ln a(s)^{1/4}\Big|_r^t$$

shows that H satisfies the estimate

$$\|H(t, r, \xi)\| \leqslant C\|R_1\| \leqslant \frac{C}{\langle\xi\rangle^\sigma}\left(\frac{1}{t}\mu(t)\right)^2. \tag{2.8}$$

Finally, we define the matrix-valued function $Q = Q(t, r, \xi)$ by

$$Q(t, r, \xi) := \sum_{k=1}^{\infty} i^k \int_r^t H(t_1, r, \xi)\, dt_1 \int_r^{t_1} H(t_2, r, \xi)\, dt_2 \cdots \int_r^{t_{k-1}} H(t_k, r, \xi)\, dt_k,$$

thus

$$V_1(t,\xi) := E_2(t,t_\xi,\xi)\big(I + Q(t,t_\xi,\xi)\big)V_{1,0}(\xi)$$

represents the solution of (2.6). Furthermore, we have

$$\big\|I + Q(t,r,\xi)\big\| \leqslant \exp\left(\int_{t_\xi}^{t} \big\|H(s,t_\xi,\xi)\big\|\,ds\right) \leqslant \exp\left(\int_{t_\xi}^{t} \frac{C(\frac{1}{s}\mu(s))^2}{\langle\xi\rangle^\sigma}\,ds\right)$$

$$\leqslant \exp\left(\frac{C\mu^2(t_\xi)}{\langle\xi\rangle^\sigma}\int_{t_\xi}^{t}\frac{1}{s^2}\,ds\right) \leqslant \exp\left(\frac{C\mu^2(t_\xi)}{\langle\xi\rangle^\sigma t_\xi}\right) = \exp\left(\frac{C}{N}\mu(t_\xi)\right).$$

Consequently, the solution of the system (2.6) in $Z_{hyp}(M,N)$ satisfies:

$$\big|V(t,\xi)\big| \leqslant C\exp\left(\frac{C}{N}\mu(t_\xi)\right)\big|V(t_\xi,\xi)\big|. \tag{2.9}$$

2.3. Conclusion

Combining the estimations (2.4), (2.5) and (2.9), we have the following lemma.

Lemma 2.2. *Let us consider the Cauchy problem*

$$v_{tt} + a(t)|\xi|^{2\sigma}v = 0, \qquad v(0,\xi) = \hat{\varphi}(\xi), \qquad v_t(0,\xi) = \hat{\psi}(\xi),$$

where $a(t)$ belongs to $L^\infty[0,T]\cap C^2(0,T]$ and its first and second order derivatives satisfy:

$$\big|a^{(k)}(t)\big| \leqslant C_k\left(\frac{1}{t}\mu(t)\right)^k, \quad k = 1,2.$$

We have the following a priori estimate

$$\left|\begin{pmatrix} |\xi|^\sigma v(t,\xi) \\ v_t(t,\xi)\end{pmatrix}\right| \lesssim \exp\big(C\mu(t_\xi)\big)\big(|\xi|^\sigma\big|\hat{\varphi}(\xi)\big| + \big|\hat{\psi}(\xi)\big|\big).$$

With the Plancherel theorem, this gives Theorem 1.1 immediately.
By a carefully calculation, Theorem 1.1 indicates the following corollary.

Corollary 2.1. *When $\mu(t) = \log(1/t)\log^{[n]}(1/t)$, $n \geqslant 2$, as stated in* (1.1)*, then there is at most an infinite loss for* (1.2)*.*

Proof. By inserting (1.1) into $\frac{\mu(t_\xi)}{t_\xi} = \frac{\langle\xi\rangle^\sigma}{N}$, we have

$$(1/t_\xi)\log(1/t_\xi)\log^{[n]}(1/t_\xi) = \frac{\langle\xi\rangle^\sigma}{N}.$$

Taking a logarithm at both sides gives:

$$\log(1/t_\xi) + \log^{[2]}(1/t_\xi) + \log^{[n+1]}(1/t_\xi) = \log\left(\frac{\langle\xi\rangle^\sigma}{N}\right).$$

Thus $\log(1/t_\xi) \leqslant \log(\frac{\langle\xi\rangle^\sigma}{N})$ and for the large ξ and small t_ξ, we have the approximation:

$$\log(1/t_\xi) \approx \log\frac{\langle\xi\rangle^\sigma}{N}.$$

Finally, we have the following estimation:

$$\exp\big(C\mu(t_\xi)\big) = \exp\big\{C\log(1/t_\xi)\log^{[n]}(1/t_\xi)\big\} \leqslant \left(\frac{\langle\xi\rangle^\sigma}{N}\right)^{C\log^{[n]}(\frac{\langle\xi\rangle^\sigma}{N})}.$$

The term $\log^{[n]}(\frac{\langle\xi\rangle^\sigma}{N})$ goes to infinity when $|\xi|$ goes to infinity. This shows the infinite loss of regularity for (1.2). □

Remark 2.1. Here the loss of regularity is independent of t and this implies an a priori estimate at any time t.

3. Proof of Theorem 1.2

In this section, we introduce the following micro-energy in the ultradistributional sense

$$E(\hat{u})(t,\xi) := \left|\hat{u}(t,\xi)\right| + \left|\partial_t\hat{u}(t,\xi)\right|.$$

According to the discussion in [11], H^∞ well-posedness indicates the existence of two non-negative constants κ and C, such that for any solution $u(t,x)$, any $t^{(1)}, t^{(2)} \in [0,T]$ and $\xi \in \mathbb{R}^N$, the micro-energy of the partial Fourier transform $\hat{u}(t,\xi)$ satisfies the estimate

$$E(\hat{u})\big(t^{(2)},\xi\big) \leqslant C\big(1+|\xi|^\sigma\big)^\kappa E(\hat{u})\big(t^{(1)},\xi\big).$$

Our aim is to show that, when $\mu(t) = \log(1/t)\log^{[n]}(1/t)$, $n \geqslant 2$, there exist solutions $\{\hat{u}_m(t,\xi)\}_m$ to the partial Fourier transformed equation of (1.3),

$$\partial_t^2\hat{u} + b^2\big(\big(\log(1/t)\big)^2\log^{[n]}(1/t)\big)|\xi|^{2\sigma}\hat{u} = 0, \tag{3.1}$$

accompanied by a sequence of frequencies $\{\xi_m\}_m$ and a sequence of time-pairs $\{(t_m^{(1)}, t_m^{(2)})\}_m$ $(0 < t_m^{(1)} < t_m^{(2)} < T)$, with $\lim_{m\to\infty}|\xi_m| = \infty$, $\lim_{m\to\infty} t_m^{(1)} = \lim_{m\to\infty} t_m^{(2)} = 0$, and $\hat{u}_m(t,\xi)$ satisfies

$$E(\hat{u}_m)\big(t_m^{(2)},\xi_m\big) \geqslant C\exp\Big(C_1\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}\,\Big)E(\hat{u}_m)\big(t_m^{(1)},\xi_m\big),$$

C and C_1 are positive constants. Clearly, this estimate shows directly the contradiction to the H^∞ well-posedness.

3.1. Derivation of an auxiliary Cauchy problem

Apply the method of coordinate transformation and define

$$s(t) := \big(\log(1/t)\big)^2\log^{[n]}(1/t), \qquad \tau(s) := -\frac{ds}{dt}\big(t(s)\big), \qquad w(s,\xi) := \sqrt{\tau(s)}\hat{u}\big(t(s),\xi\big),$$

then we have the following system

$$\partial_s^2 w(s,\xi) + \lambda(s,\xi)b^2(s)w(s,\xi) = 0, \quad (s,\xi) \in \big[s(T),\infty\big) \times \mathbb{R}^N,$$

with

$$\lambda(s,\xi) = \lambda_1(s,\xi) + \lambda_2(s) = \frac{|\xi|^{2\sigma}}{\tau^2(s)} + \frac{\theta(s)}{b^2(s)\tau^2(s)}, \quad \theta(s) = \frac{(\partial_s\tau)^2 - 2\tau\partial_s^2\tau}{4}.$$

A simple calculation leads to the following facts:

$$\tau(s) \approx 2\sqrt{s}\sqrt{\log^{[n-1]}\sqrt{s}}\exp\left(\sqrt{\frac{s}{\log^{[n-1]}\sqrt{s}}}\right),$$

$$\theta(s) \approx -\frac{1}{4}\frac{1}{(\log(1/t)\log^{[n]}(1/t))^2}\bigg|_{t=t(s)}.$$

As a result, $\lim_{s\to\infty}\lambda_2(s) = 0$.

Let λ_0 be a positive real number and define $s_\xi = s_\xi(\lambda_0)$ as the solution of $\lambda(s,\xi) = \lambda_0$, then after a simple calculation, one knows that $s_\xi \sim (\log|\xi|^\sigma)^2\log^{[n]}|\xi|^\sigma$. Next, we introduce some estimates which are of vital importance in the instability argument.

Lemma 3.1. *For $0 \leqslant \rho \leqslant \log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}$ and large s_ξ, the following estimates hold,*

$$|\Delta_\rho\lambda_1| \lesssim \lambda_1(s_\xi,\xi)\Big(\sqrt{\log^{[n]}|\xi|^\sigma}\,\Big)^{-1}, \qquad |\Delta_\rho\lambda_2| \lesssim \lambda_2(s_\xi)\Big(\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}\,\Big)^{-1},$$

with $\Delta_\rho\lambda_1 := \lambda_1(s_\xi,\xi) - \lambda_1(s_\xi - \rho,\xi)$, $\Delta_\rho\lambda_2 := \lambda_2(s_\xi) - \lambda_2(s_\xi - \rho)$. Combining the two estimates, one obtains $|\Delta_\rho\lambda| \lesssim \lambda(s_\xi,\xi)(\sqrt{\log^{[n]}|\xi|^\sigma})^{-1}$ with $\Delta_\rho\lambda := \lambda(s_\xi,\xi) - \lambda(s_\xi - \rho,\xi)$.

Proof. Notice the fact that $s_\xi \sim (\log|\xi|^\sigma)^2\log^{[n]}|\xi|^\sigma$, then for $0 \leqslant \rho \leqslant \log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}$,

$$|\Delta_\rho \lambda_1| = \left| |\xi|^{2\sigma}(s_\xi - \rho)^{-1}\big(\log^{[n-1]}\sqrt{s_\xi - \rho}\big)^{-1}\exp\left(-2\sqrt{\frac{s_\xi - \rho}{\log^{[n-1]}\sqrt{s_\xi - \rho}}}\right) - \lambda_1(s_\xi, \xi)\right|$$
$$\sim \lambda_1(s_\xi, \xi)\left|(1 - \rho/s_\xi)^{-1}\exp\left(-2\sqrt{\frac{s_\xi}{\log^{[n-1]}\sqrt{s_\xi}}}(\sqrt{1-\rho/s_\xi} - 1)\right) - 1\right|$$
$$\lesssim \lambda_1(s_\xi, \xi)\sqrt{\log^{[n]}|\xi|^\sigma},$$
$$|\Delta_\rho \lambda_2| \sim \left|(s_\xi - \rho)^{-1}\big(\log^{[n-1]}\sqrt{s_\xi - \rho}\big)^{-1} - \lambda_2(s_\xi)\right| \lesssim \lambda_2(s_\xi)\Big(\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}\Big)^{-1}. \qquad \square$$

3.2. Instability argument and application of Floquet theory

We are interested in the fundamental solution $X = X(s, s_0)$ of the Cauchy problem

$$\frac{\partial}{\partial s}X = A(s)X = \begin{pmatrix} 0 & -\lambda_0 b^2(s) \\ 1 & 0 \end{pmatrix} X, \qquad X(s_0, s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $X(s_0 - 1, s_0)$ is independent of $s_0 \in \mathbb{N}$ since $b(s)$ is 1-periodic. For this periodic problem, we have the following instability conclusion: (see [10]).

Lemma 3.2 *(Floquet theory). Let $b(s) \in C^2(\mathbb{R})$ be a non-constant, positive, 1-periodic function, then there exists a positive real number $\lambda_0 > 0$ such that λ_0 belongs to an interval of instability for $\partial_s^2 w + \lambda_0 b^2(s) w = 0$, that is, $X(s_0 - 1, s_0)$ has eigenvalues μ_0 and μ_0^{-1} satisfying $|\mu_0| > 1$.*

Set $X(s_\xi - 1, s_\xi) = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, then Lemma 3.3 implies that the eigenvalues of this matrix are μ_0 and μ_0^{-1}. Hence, $a_{11} + a_{22} = \mu_0 + \mu_0^{-1}$, which gives $|a_{11} - \mu_0| + |a_{22} - \mu_0| \geqslant |\mu_0 - \mu_0^{-1}|$. From this estimate it follows $\max\{|a_{11} - \mu_0|, |a_{22} - \mu_0|\} \geqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$. Without loss of generality, we assume that $|a_{11} - \mu_0| \geqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, then we have $|a_{22} - \mu_0^{-1}| \geqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$. Now we consider the following equation with an integer $m \geqslant 1$:

$$\partial_s^2 w + \frac{1}{4}\mathscr{B}(s, s_\xi; m)|\xi|^{2\sigma} b^2(s_\xi + s) w = 0,$$

with

$$\mathscr{B}(s, s_\xi; m) := (s_\xi - m + s)^{-1}\big(\log^{[n-1]}\sqrt{s_\xi - m + s}\big)^{-1}\exp\left(-2\sqrt{\frac{s_\xi - m + s}{\log^{[n-1]}\sqrt{s_\xi - m + s}}}\right).$$

Let $X_m(s, s_1)$ be the solution of the associated first order system,

$$\frac{\partial}{\partial s}X_m(s, s_1) = \begin{pmatrix} 0 & -\frac{1}{4}\mathscr{B}(s, s_\xi; m)|\xi|^{2\sigma} b^2(s_\xi + s) \\ 1 & 0 \end{pmatrix} X_m(s, s_1),$$
$$X_m(s_1, s_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Next we give a collection of properties concerned with the fundamental solutions $X(t, s)$ and $X_m(t, s)$, which will help us achieve the aim.

Lemma 3.3. *For an integer m, $1 \leqslant m \leqslant \log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}$, the following statements hold:*

(a) $\max_{s, s_1 \in [-1,0]}\|X_m(s, s_1)\| \leqslant \exp(C\lambda_0)$.
(b) $\|X_m(-1, 0) - X(s_\xi - 1, s_\xi)\| \lesssim (\sqrt{\log^{[n]}|\xi|^\sigma})^{-1}$.
(c) $\|X_{m+1}(-1, 0) - X_m(-1, 0)\| \lesssim (\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma)^{-1}$.

Proof. (a) The first statement follows when we apply the matrizant method used in Section 2, that is, with $c = \max_{s \in \mathbb{R}}\{b(s)\}$ and ε sufficiently small,

$$\max_{s, s_1 \in [-1,0]}\|X_m(s, s_1)\| \leqslant \exp\big(1 + c^2|\lambda(s_\xi - m + s, \xi) - \lambda(s_\xi, \xi) + \lambda_0|\big)$$
$$\leqslant \exp\big(1 + c^2(1 + \varepsilon)\lambda_0\big) \leqslant \exp(C\lambda_0).$$

(b) Choose $s_\xi \in \mathbb{N}_+$, then $X(s_\xi + s, s_\xi) = X(s, 0)$ since $b(s)$ is 1-periodic. Actually we have

$$\frac{\partial}{\partial s}(X_m - X) = \begin{pmatrix} 0 & -\lambda(s_\xi, \xi)b^2(s) \\ 1 & 0 \end{pmatrix}(X_m - X) + \begin{pmatrix} 0 & \Delta_{m-s}\lambda b^2(s) \\ 0 & 0 \end{pmatrix} X_m$$

with initial data $X_m(0,0) - X(0,0) = 0$. $\Delta_{m-s}\lambda$ denotes as usual $\lambda(s_\xi, \xi) - \lambda(s_\xi - m + s, \xi)$. Lemma 3.2 gives

$$|\Delta_{m-s}\lambda| \lesssim \lambda_0\Big(\sqrt{\log^{[n]}|\xi|^\sigma}\Big)^{-1}.$$

Taking this into account, we obtain

$$\big\|X_m(s,0) - X(s,0)\big\| \lesssim \int_0^s \lambda_0 \big\|X_m(r,0) - X(r,0)\big\|\,dr + \int_0^s \lambda_0\Big(\sqrt{\log^{[n]}|\xi|^\sigma}\Big)^{-1}\big\|X_m(r,0)\big\|\,dr.$$

Now we apply Gronwall's inequality and obtain

$$\big\|X_m(-1,0) - X(-1,0)\big\| \lesssim \lambda_0\Big(\sqrt{\log^{[n]}|\xi|^\sigma}\Big)^{-1}.$$

(c) As in (b), we consider the following system,

$$\frac{\partial}{\partial s}(X_{m+1} - X_m) = \begin{pmatrix} 0 & -\lambda(s_\xi - m - 1 + s, \xi)b^2(s) \\ 1 & 0 \end{pmatrix}(X_{m+1} - X_m) + \begin{pmatrix} 0 & \Delta_1\lambda b^2(s) \\ 0 & 0 \end{pmatrix} X_m$$

with initial data $X_{m+1}(0,0) - X_m(0,0) = 0$. And $\Delta_1\lambda$ denotes $\lambda(s_\xi - m + s, \xi) - \lambda(s_\xi - m - 1 + s)$. Following the proof of Lemma 3.2, we obtain $|\Delta_1\lambda| \lesssim \lambda_0(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma)^{-1}$. By a similar argument as in (b), we arrive at the conclusion. $\square$

Lemma 3.3(b) and (c) indicate $\|X_m(-1,0) - X(s_\xi - 1, s_\xi)\| \lesssim (\sqrt{\log^{[n]}|\xi|^\sigma})^{-1}$, thus the matrix $X_m(-1,0)$ with $\det X_m(-1,0) = 1$ has eigenvalues μ_m and μ_m^{-1} satisfying

$$|\mu_m - \mu_0| \lesssim \Big(\sqrt{\log^{[n]}|\xi|^\sigma}\Big)^{-1} \leqslant \epsilon$$

for any given positive ϵ. By choosing $\epsilon \leqslant (|\mu_0| - 1)/2$, we have $|\mu_m| \geqslant (|\mu_0| + 1)/2 \geqslant 1 + \epsilon$, which indicates that the eigenvalues μ_m and μ_m^{-1} are uniformly distinct for every m. Let $X_m(-1,0)$ be as $\left(\begin{smallmatrix} x_{11}(m) & x_{12}(m) \\ x_{21}(m) & x_{22}(m)\end{smallmatrix}\right)$ and recall that $X(-1,0) = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)$, then we have

$$\big|x_{11}(m) - \mu_m\big| \geqslant |a_{11} - \mu_0| - \big(|x_{11}(m) - a_{11}| + |\mu_0 - \mu_m|\big) \geqslant |\mu_0 - \mu_0^{-1}|/4.$$

Analogously, the estimate $|x_{22}(m) - \mu_m^{-1}| \geqslant |\mu_0 - \mu_0^{-1}|/4$ holds. Lemma 3.3(c) tells us that

$$\big|x_{ij}(m) - x_{ij}(m-1)\big| \lesssim \big(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma\big)^{-1}, \quad i, j = 1, 2.$$

From these inequalities, we conclude immediately

$$|\mu_m - \mu_{m-1}| \lesssim \big(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma\big)^{-1}.$$

Lemma 3.4. *Let m_0 be the largest integer satisfying $1 \leqslant m \leqslant \log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}$. Assume $w(s,\xi)$ is solution of the equation*

$$\partial_s^2 w(s,\xi) + \lambda(s,\xi)b^2(s)w(s,\xi) = 0, \quad (s,\xi) \in \big[s(T), \infty\big) \times \mathbb{R}^N \tag{3.3}$$

with initial data

$$w(s_\xi - 1, \xi) = 1, \qquad \partial_s w(s_\xi - 1, \xi) = \frac{x_{12}(1)}{\mu_1 - x_{11}(1)},$$

then there exist constants C and C_1, such that

$$\big|\partial_s w(s_\xi - m_0 - 1, \xi)\big| + \big|w(s_\xi - m_0 - 1, \xi)\big| \geqslant C\exp\Big(C_1 \log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}\Big).$$

Proof. The function $w(s_\xi - m_0 + s, \xi)$ satisfies $\partial_s^2 w + \lambda(s_\xi - m_0 + s, \xi)b^2(s_\xi + s)w = 0$, and

$$\begin{pmatrix} \partial_s w(s_\xi - m_0 - 1, \xi) \\ w(s_\xi - m_0 - 1, \xi) \end{pmatrix} = X_{m_0}(-1,0)X_{m_0-1}(-1,0)\cdots X_1(-1,0)\begin{pmatrix} \partial_s w(s_\xi - 1, \xi) \\ w(s_\xi - 1, \xi) \end{pmatrix}.$$

Clearly, a diagonalizer for $X_m(-1,0)$ is the matrix

$$B_m = \begin{pmatrix} \frac{x_{12}(m)}{\mu_m - x_{11}(m)} & 1 \\ 1 & \frac{x_{21}(m)}{\mu_m^{-1} - x_{22}(m)} \end{pmatrix}.$$

Since $\det X_m(-1,0) = 1$ and the trace of $X_m(-1,0)$ is $\mu_m + \mu_m^{-1}$, then we have

$$\det B_m = \frac{\mu_m - \mu_m^{-1}}{\mu_m^{-1} - x_{22}(m)}.$$

Lemma 3.2 and $|\mu_m| \geqslant 1+\epsilon$ indicate that, for $1 \leqslant m \leqslant \log|\xi|^\sigma \sqrt{\log^{[n]}|\xi|^\sigma}$,

$$|\det B_m| \geqslant C > 0, \qquad \|B_m\| \leqslant C, \qquad \|B_m^{-1}\| \leqslant C.$$

For $m \geqslant 2$, Lemma 3.4 gives

$$\bigl|x_{ij}(m) - x_{ij}(m-1)\bigr| \leqslant C\bigl(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma\bigr)^{-1},$$

with a positive constant C. Accordingly, we obtain

$$\bigl\|B_m^{-1}B_{m-1} - I\bigr\| = \bigl\|B_m^{-1}(B_{m-1} - B_m)\bigr\| \leqslant C\bigl(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma\bigr)^{-1}.$$

Denote G_m as $B_m^{-1}B_{m-1} - I$, then

$$X_{m_0}(-1,0)X_{m_0-1}(-1,0)\cdots X_1(-1,0) = B_{m_0}\begin{pmatrix} \mu_{m_0} & 0 \\ 0 & \mu_{m_0}^{-1} \end{pmatrix}(I+G_{m_0})\cdots(I+G_2)\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix}B_1^{-1}.$$

We shall show that the $(1,1)$ element y_{11} of the matrix

$$\begin{pmatrix} \mu_{m_0} & 0 \\ 0 & \mu_{m_0}^{-1} \end{pmatrix}(I+G_{m_0})\cdots(I+G_2)\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix}$$

can be estimated from below by $C_0\exp(C_1\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma})$. In fact, the above matrix can be represented by

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} \prod_{k=1}^{m_0}\mu_k & 0 \\ 0 & \prod_{k=1}^{m_0}\mu_k^{-1} \end{pmatrix} + M_1 + \cdots + M_{m_0},$$

where M_ℓ ($\ell = 1,\dots,m_0$) is the sum of terms containing exactly ℓ of the matrices G_k, $k = 1,\dots,m_0$. Recall that $\|G_k\| \leqslant C(\log|\xi|^\sigma\log^{[n]}|\xi|^\sigma)^{-1}$, $k = 1,\dots,m_0$, then it follows

$$\|M_\ell\| \leqslant C^\ell \prod_{k=1}^{m_0}|\mu_k| \begin{pmatrix} m_0 \\ \ell \end{pmatrix} \bigl(\log|\xi|^\sigma \log^{[n]}|\xi|^\sigma\bigr)^{-\ell}.$$

Consequently,

$$\left| y_{11} - \prod_{k=1}^{m_0}\mu_k \right| \leqslant \prod_{k=1}^{m_0}|\mu_k|\Bigl(\bigl(1 + C\bigl(\log|\xi|^\sigma\log^{[n]}|\xi|^\sigma\bigr)^{-1}\bigr)^{\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}} - 1\Bigr).$$

Notice that $\lim_{|\xi|\to\infty}(1 + C(\log|\xi|^\sigma\log^{[n]}|\xi|^\sigma)^{-1})^{\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}} = 1$, then for any positive real number ε, as long as $|\xi|$ is large enough, we have

$$\left| |y_{11}| - \prod_{k=1}^{m_0}|\mu_k| \right| \leqslant \varepsilon \prod_{k=1}^{m_0}|\mu_k|.$$

It follows from $|\mu_k - \mu_0| \leqslant \epsilon$ that

$$|y_{11}| \geqslant (1-\varepsilon)\prod_{k=1}^{m_0}|\mu_k| \geqslant (1-\varepsilon)\bigl(|\mu_0| - \epsilon\bigr)^{\log|\xi|^\sigma\sqrt{\log^{[n]}|\xi|^\sigma}}.$$

On the other hand, $|y_{ij}| \leqslant \varepsilon\prod_{k=1}^{m_0}|\mu_k|$, with ε arbitrarily small for $(i,j) \neq (1,1)$. Thus the statement of the lemma follows directly. $\square$

3.3. Estimate of the micro-energy

Now we take a sequence $\{\xi_m\}_m$ satisfying $|\xi_m| \to \infty$ as $m \to \infty$. For large $|\xi_m|$, let $w_m(s,\xi)$ be the solution of (3.3) with $\xi = \xi_m$ and define two zero sequences

$$t_m^{(1)} := t(s_{\xi_m} - 1), \qquad t_m^{(2)} := t(s_{\xi_m} - m_0 - 1).$$

Keep in mind the transformation

$$\hat{u}_m(t,\xi_m) = \left(\sqrt{\tau\big(s(t)\big)}\right)^{-1} w_m\big(s(t),\xi_m\big),$$

then for $t \in (0,T]$, we conclude

$$\left|\partial_s w_m\big(s(t),\xi_m\big)\right| \leqslant \frac{1}{2}\partial_s\tau\big(s(t)\big)\left(\sqrt{\tau\big(s(t)\big)}\right)^{-1}\left|\hat{u}_m(t,\xi_m)\right| + \left(\sqrt{\tau\big(s(t)\big)}\right)^{-1}\left|\partial_t\hat{u}_m(t,\xi_m)\right|.$$

Hence,

$$\begin{aligned} E(w_m)\big(s(t),\xi_m\big) &\leqslant \sqrt{\tau\big(s(t)\big)}\left(1 + \frac{1}{2}\partial_s\tau\big(s(t)\big)\big(\tau\big(s(t)\big)\big)^{-1}\right)\left|\hat{u}_m(t,\xi_m)\right| \\ &\quad + \left(\sqrt{\tau\big(s(t)\big)}\right)^{-1}\left|\partial_t\hat{u}_m(t,\xi_m)\right| \lesssim \sqrt{\tau\big(s(t)\big)}E(\hat{u}_m)(t,\xi_m), \\ E(\hat{u}_m)(t,\xi_m) &\lesssim \sqrt{\tau\big(s(t)\big)}E(w_m)\big(s(t),\xi_m\big). \end{aligned}$$

Summarizing the above discussions, we have

$$\begin{aligned} E(\hat{u}_m)\big(t_m^{(2)},\xi_m\big) &\geqslant C\left(\sqrt{\tau\big(s\big(t_m^{(2)}\big)\big)}\right)^{-1}\left(\left|w_m\big(s\big(t_m^{(2)}\big),\xi_m\big)\right| + \left|\partial_s w_m\big(s\big(t_m^{(2)}\big),\xi_m\big)\right|\right) \\ &\geqslant C\left(\sqrt{\tau\big(s\big(t_m^{(2)}\big)\big)}\right)^{-1}\left(\sqrt{\tau\big(s\big(t_m^{(1)}\big)\big)}\right)^{-1}\exp\left(C_1\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}\right)E(\hat{u}_m)\big(t_m^{(1)},\xi_m\big) \\ &\geqslant C\exp\left(C_1\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}\right)E(\hat{u}_m)\big(t_m^{(1)},\xi_m\big), \end{aligned}$$

where C and C_1 are used as universal positive constants. The last inequality holds since $\tau(s(t_m^{(i)}))$, $i = 1,2$, are essentially equivalent to

$$\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}\sqrt{\log^{[n-1]}\left(\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}\right)}\exp\left(\frac{\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma}}{\sqrt{\log^{[n-1]}(\log|\xi_m|^\sigma\sqrt{\log^{[n]}|\xi_m|^\sigma})}}\right).$$

These complete our proof.

Remark 3.1. In fact, our counter-example is constructed in ultradistribution space. Whether this proof can be extended to the Sobolev space is still an open problem.

4. Proof of Theorem 1.3

In this section, we discuss about the optimality of our estimates in Theorem 1.1. For $\sigma \in (0,1]$, we consider the following strictly hyperbolic Cauchy problems in $[0,T] \times \mathbb{R}$:

$$\partial_t^2 u_k + b(t)\left(-\partial_x^2\right)^\sigma u_k = 0, \qquad u_k(0,x) = u_{0,k}(x), \qquad \partial_t u_k(0,x) = u_{1,k}(x). \tag{4.1}$$

Actually, Theorem 1.1 leads to the following corollary.

Corollary 4.1. *Define $F(s)$ as $\mu(s)/s$ and $\{b_k\}_k$ satisfy all the assumptions in Theorem 1.1 independent of k. If $u_{0,k} \in H^s(\mathbb{R})$, $u_{1,k} \in H^{s-\sigma}(\mathbb{R})$, and P is a fixed appropriate positive integer, then there exists a sequence of solutions $\{u_k\}_k$ belonging to the following function spaces*:

$$C\big([0,T];\exp\big(c_1\mu\big(F^{-1}\big(2^P/(1-\Delta)^{\sigma/2}\big)\big)\big)H^s(\mathbb{R})\big) \cap C^1\big([0,T];\exp\big(c_1\mu\big(F^{-1}\big(2^P/(1-\Delta)^{\sigma/2}\big)\big)\big)H^{s-\sigma}(\mathbb{R})\big)$$

with a positive constant c_1. Moreover, F^{-1} denotes the inverse function of F.

With the discussion in Section 3 about *Floquet theory*, we introduce an important lemma for our purpose.

Lemma 4.1. *If $w(s)$ is the solution of $\partial_s^2 w + \lambda_0 b^2(s) w = 0$ with initial data $w(0) = 1$, $\partial_s w(0) = 0$. Then for every sufficiently large positive $M \in \mathbb{N}$, the solution satisfies $|w(\pm M)| \sim \mu_0^M$. ($b(s)$, λ_0 and μ_0 are given in Lemma* 3.2.)

Proof. Using the fundamental solution $X(-1,0)$, we have

$$\begin{pmatrix} \partial_s w(-M) \\ w(-M) \end{pmatrix} = \big(X(-1,0)\big)^M \begin{pmatrix} \partial_s w(0) \\ w(0) \end{pmatrix}.$$

The matrix

$$B := \begin{pmatrix} \frac{a_{12}}{\mu_0 - a_{11}} & 1 \\ 1 & \frac{a_{21}}{\mu_0^{-1} - a_{22}} \end{pmatrix}$$

is a diagonalizer for $X(-1,0)$, which means

$$X(-1,0)B = B \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}.$$

Since $\det X(-1,0) = 1$ and the trace of $X(-1,0)$ is $\mu_0 + \mu_0^{-1}$, we obtain

$$\det B = \frac{\mu_0 - \mu_0^{-1}}{\mu_0^{-1} - a_{22}}.$$

A straightforward calculation leads to

$$\begin{pmatrix} \partial_s w(-M) \\ w(-M) \end{pmatrix} = \frac{1}{\det B} \begin{pmatrix} \frac{a_{12}(\mu_0^{-M} - \mu_0^M)}{\mu_0 - a_{11}} \\ -\mu_0^M + \frac{a_{12} a_{21} \mu_0^{-M}}{(\mu_0 - a_{11})(\mu_0^{-1} - a_{22})} \end{pmatrix}.$$

Since $|\mu_0 - a_{11}| \geqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, $|\mu_0^{-1} - a_{22}| \geqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, and taking into consideration $\|X(-1,0)\| \leqslant C$, we see that $|\det B| \sim 1$. When M is sufficiently large, μ_0^M becomes the dominating part, which gives $w(-M) \sim \mu_0^M$. As for $w(M)$, we apply the properties of $X(1,0)$ and arrive at the result. □

Remark 4.1. For the case when $|\mu_0 - a_{11}| \leqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, $|\mu_0^{-1} - a_{22}| \leqslant \frac{1}{2}|\mu_0 - \mu_0^{-1}|$, we can choose

$$B := \begin{pmatrix} \frac{a_{12}}{\mu_0^{-1} - a_{11}} & 1 \\ 1 & \frac{a_{21}}{\mu_0 - a_{22}} \end{pmatrix}$$

as the diagonalizer and let $w(0) = 0$, $\partial_s w(0) = 1$.

Step 1: Introduction of auxiliary sequences
We define a sequence of intervals $\{I_k\}_k$ by

$$I_k = [t_k - \rho_k/2, t_k + \rho_k/2],$$

and choose the following sequences with $\theta_0 := \lambda_0^{\frac{1}{\sigma}}$

$$\{\rho_k\}_k = \big\{2^{-P+2}\sqrt{\theta_0}\, t_k\big[\mu(t_k)\big]/\mu(t_k)\big\}_k, \qquad \big\{h_k = 2^P(\sqrt{\theta_0})^{-1}\mu(t_k)t_k^{-1}\big\}_k.$$

Remark 4.2. We require that $\{t_k\}_k$ be an appropriate null sequence, such that the sequence $\{\rho_k\}_k$ tends to 0, while $\{\sqrt{\theta_0}\, h_k\}_k \in \mathbb{N}$ tends to $+\infty$. Furthermore, we have $h_k \rho_k/2 \in \mathbb{N}_+$.

Next we construct a family of coefficients $\{b_k = b_k^2(t)\}_k$ which are defined by

$$b_k(t) = \begin{cases} 1, & t \in [0,T] \setminus I_k; \\ b(h_k(t - t_k)), & t \in I_k. \end{cases}$$

Clearly, the definition of $b_k(t)$ indicates:

$$0 < b_0 \leqslant \inf_{t \in [0,T]} b_k(t) \leqslant \sup_{t \in [0,T]} b_k(t) \leqslant b_1 < \infty,$$

where the constants b_0 and b_1 are independent of k. Straightforward calculations show that, the coefficients b_k satisfy all assumptions in Theorem 1.1.

Step 2: At least a μ-loss

Let $\chi = \chi(r) \in [0,1]$ be a cut-off function from $C_0^\infty(\mathbb{R})$, where $\chi \equiv 1$ for $|r| \leqslant 1$ and $\chi \equiv 0$ for $|r| \geqslant 2$. Then we choose for large k the following data:

$$u_{0,k}(x) = \exp(ih_k\sqrt{\theta_0}x)\chi\left(\frac{x}{(\mu(t_k))^2 P_k}\right), \qquad u_{1,k}(x) = 0 \quad \text{for all } x \in \mathbb{R},$$

where

$$P_k = \frac{2\pi}{h_k\sqrt{\theta_0}} \sim t_k\big(\mu(t_k)\big)^{-1}.$$

Then for $s \in \mathbb{N}_+$, the norm $\|u_{0,k}\|_{H^s(\mathbb{R})}$ can be estimated in the following way:

$$\|u_{0,k}\|_{H^s(\mathbb{R})} \leqslant C\big(h_k^s + 1\big)\mu(t_k)\sqrt{P_k}. \tag{4.2}$$

Now we study the family of Cauchy problems

$$\partial_t^2 u_k + b^2\big(h_k(t-t_k)\big)\big(-\partial_x^2\big)^\sigma u_k = 0, \qquad u_k(t_k, x) = u_{0,k}(x), \qquad \partial_t u_k(t_k, x) = 0, \quad t \in [t_k - \rho_k/2, t_k + \rho_k/2].$$

If x is taken from $\{|x| \leqslant P_k\}$ on $t = t_k + \rho_k/2$, then the solution $u_k(t_k + \rho_k/2, x)$ will be influenced by the data on the set $\{|x| \leqslant P_k + \rho_k/2\}$. In this set, we have $u_{0,k}(x) = \exp(ih_k\sqrt{\theta_0}x)$. We apply the transformation $s = h_k^\sigma(t - t_k)$, and define $v_k(s,x) := u_k(t(s), x)$, then we get:

$$\partial_s^2 v_k - h_k^{-2\sigma} b^2(s)\partial_x^2 v_k = 0, \qquad v_k(0,x) = u_{0,k}(x), \qquad \partial_s v_k(0,x) = 0, \quad s \in [-h_k\rho_k/2, h_k\rho_k/2].$$

In fact, we have a unique solution in the form $v_k(s,x) = u_{0,k}(x)w(s)$, where $w = w(s)$ satisfies

$$\partial_s^2 w(s) + \lambda_0 b^2(s)w(s) = 0, \qquad w(0) = 1, \qquad \partial_s w(0) = 0, \quad s \in [-h_k\rho_k/2, h_k\rho_k/2].$$

Transforming back to $u_k(t,x)$, we arrive at

$$u_k(t_k + \rho_k/2, x) = \exp(ih_k\sqrt{\theta_0}x)w(\rho_k h_k/2), \qquad u_k(t_k, x) = \exp(ih_k\sqrt{\theta_0}x)w(0),$$

where $|w(\rho_k h_k/2)| \sim |\mu_0|^{\rho_k h_k/2}$. According to the definition of pseudodifferential operators, one has

$$\big\|\exp\big(-c_1\mu\big(F^{-1}\big(\langle D_x\rangle/2^P\big)\big)\big)\langle D_x\rangle^s u_k(t_k + \rho_k/2, \cdot)\big\|_{\{|x|\leqslant P_k\}} \sim \exp\big(-c_1\mu(t_k)\big)\big(h_k^s + 1\big)\sqrt{P_k}|\mu_0|^{\rho_k h_k/2}. \tag{4.3}$$

Denote $t_k^{(1)} = t_k$, $t_k^{(2)} = t_k + \rho_k/2$, then we get the following estimate from (4.2) and (4.3).

$$\begin{aligned}\big\|\exp\big(-c_1\mu\big(F^{-1}\big(\langle D_x\rangle/2^P\big)\big)\big)\langle D_x\rangle^s u_k\big(t_k^{(2)}, \cdot\big)\big\|_{L^2(\mathbb{R})} &\geqslant \big\|\exp\big(-c_1\mu\big(F^{-1}\big(\langle D_x\rangle/2^P\big)\big)\big)\langle D_x\rangle^s u_k\big(t_k^{(2)}, \cdot\big)\big\|_{\{|x|\leqslant P_k\}} \\ &\geqslant C_k\big\|u_k\big(t_k^{(1)}, \cdot\big)\big\|_{H^s(\mathbb{R})}.\end{aligned}$$

Since $\rho_k h_k = 4[\mu(t_k)]$, so a sufficiently small c_1 leads to the fact $\sup_k C_k = +\infty$ and moreover, c_1 is independent of k. This shows that the *μ-loss of derivatives* really appears.

Remark 4.3. Periodic functions are very useful in the construction of coefficients for instability arguments. When we consider the problems in $[0,T] \times \mathbb{R}^N$, a compact support with respect to the initial data is needed to ensure the H^s-boundedness of the periodic functions. And with the property of finite propagation speed for $\sigma \in (0,1]$, we are able to determine the compact support of the solution at time t and find the precise representation of the solution in an interval from this support. However, when $\sigma > 1$, how to construct a counter-example remains open due to the lack of this property. For more open problems, please see [4].

5. Conclusion and open problems

Our model generalizes the wave operator and p-evolution operators, and shows essentially the common properties with regards to infinite loss and μ-loss. Combining with the case of finite loss in [6], we have formed a complete view on the phenomenon of loss engendered by oscillating coefficients. There are of course more open problems in this field.

First, in Theorem 1.1, we obtain the *at most* loss of regularity and show the infinite loss does appear for the special $\mu(t)$. However, the lower bound of the loss is not obtained yet. Second, as the previous remark stated, the proof of Theorem 1.2 is made in the ultradistributional sense. Whether we could find a counter-example in Sobolev space for such $\mu(t)$ is still unknown. Third, due to lack of finite propagation speed, how to construct a counter-example for general $\sigma > 1$ in $H^s(\mathbb{R})$ is open. Attempt in this respect was made in $H^s(\mathbb{T})$ by [6].

Besides, it is very interesting to consider the more complicated coefficients, for instance, degeneration/oscillation coupled coefficients, how to construct the counter-examples with explicit representation to show the exact μ-loss is still open, due to the lack of appropriate auxiliary functions. There are still lots of work to do.

Acknowledgments

The authors wish to thank Professor Michael Reissig (TU Bergakademie Freiberg, Germany) for his constructive suggestion during the research. And we express our deep gratitude to the referees for their careful reading and helpful remarks.

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